Tunneling in real time

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The aim:

Show that vacuum tunneling rates can be computed in Minkowski space-time

Provide an optical theorem interpretation of vacuum decay

The novelty:

There was no understanding of the behaviour of fluctuation determinants under rotations of the time-contour

Tunneling calculations had not been connected with the optical theorem

Effective potential at the minimum fixed by vacuum normalization

The plan:

Vacuum-to-vacuum transitions and decay rates

Picard-Lefschetz theory and complex saddle points

Minkowski bounce and fluctuation determinant

Why do we care?

Why do we care?

Tunneling calculations relevant in solid state and particle physics

Consistency check of usual assumptions

Development of functional techniques for rotated time contours

New insights?

Vacuum-to-vacuum transitions and decay rates

Vacuum transition amplitudes

$$Z[T] = \langle q|e^{-iHT(1-i\epsilon)}|q\rangle = \sum_n |\langle q|n\rangle|^2 e^{-iE_nT(1-i\epsilon)} \sim |\langle q|0\rangle|^2 e^{-iE_0T}$$

Analytic continuation to Euclidean time [Callan, Coleman]

$$Z_E[T_E] = \langle q|e^{-HT_E}|q\rangle = \sum_n |\langle q|n\rangle|^2 e^{-E_nT} \sim |\langle q|0\rangle|^2 e^{-E_0T_E}$$

Decay rate per unit volume of vacuum state

$$\gamma = \mathrm{Im}\left(-\frac{2}{V}E_0\right) = \mathrm{Im}\left(-\frac{2i}{VT}\log Z[T]\right) = \mathrm{Im}\left(\frac{2}{VT_E}\log Z_E[T_E]\right)$$

Vacuum transition amplitudes

$$Z[T] = \langle q| e^{-iHT(1-i\epsilon)} |q\rangle = \int [d\phi] e^{iS[\phi]}$$

$$Z_E[T_E] = \langle q| e^{-HT_E} |q\rangle = \int [d\phi] e^{-S_E}$$

How come Z_E is complex, when Euclidean path integral is real?

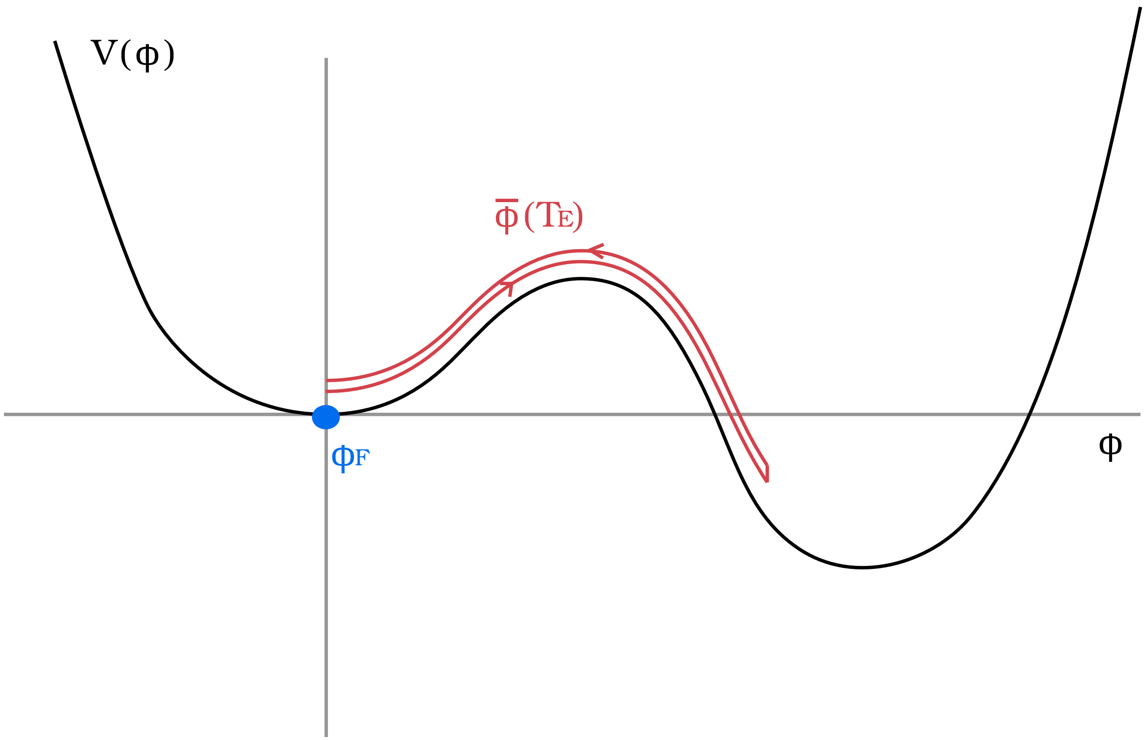
Is it really true that, with path integral definition, $Z[T] = Z_E[iT]$??

Do the tunneling rates agree?

The Euclidean calculation

Expand path integral around saddle point $\bar{\phi}$ (the “bounce”) [Callan & Coleman]

$$\left.\frac{\delta S_E}{\delta \phi}\right|_{\phi=\bar{\phi}} = 0, \qquad \lim_{t\to\pm T/2} \bar{\phi} = \phi_F$$



The Euclidean calculation

$$Z_E[T_E] = e^{-Z_E^{1\ \text{bounce}}[T_E]}$$

$$Z_E^{1\ \text{bounce}}[T_E] = \int_0 [d\phi] e^{-S_E[\bar{\phi}+\phi]} = e^{-S_E[\bar{\phi}]} \int_0 [d\phi] e^{-\frac{1}{2}\phi \mathfrak{M}_E \phi} \quad (= e^{-S_E[\bar{\phi}]} (\det \mathfrak{M}_E)^{-1/2} ?)$$

$\mathfrak{M}_E$ has: 1 negative eigenvalue, 4 zero modes: divergent integral?

Trade zero modes by integration over center of bounce

Define integration on negative mode direction by analytic continuation
This gives desired imaginary part!

$$Z_E^{1\ \text{bounce}}[T_E] = \frac{1}{2} Z_E^{\text{Gaussian, 1 bounce}}[T_E] = e^{-S_E[\bar{\phi}]} \sqrt{\frac{S_E[\bar{\phi}]}{2\pi}} V T_E \quad \left(-\frac{i}{2} |\det{}' \mathfrak{M}_E|^{-1/2} \right)$$

$$\gamma = \text{Im} \left(\frac{2}{V T_E} \log Z_E[T_E] \right) = \text{Im} \left(-\frac{2}{V T_E} Z_E^{1\ \text{bounce}}[T_E] \right) = e^{-S_E[\bar{\phi}]} \sqrt{\frac{S_E[\bar{\phi}]}{2\pi}} \; |\det{}' \mathfrak{M}_E|^{-1/2}$$

Connection to the effective action

$$Z_E[T,J] = \int [d\phi] e^{-S_E[\phi]+J\phi} \equiv e^{W[J]}$$

$$\bar{\phi} = \frac{\delta W}{\delta J} = \frac{1}{Z[J]} \int [d\phi] \phi e^{-S_E + J\phi}$$

$$\Gamma_E[T,\overline{\phi}] = W[J] - J\overline{\phi}$$

$$e^{-\Gamma_E[T,\overline{\phi}]} = \int_0 [d\phi] e^{-S[\overline{\phi}+\phi] - \Gamma'_E[T,\overline{\phi}]\phi_j}$$

$$\langle F | e^{-HT_E} | F \rangle = \int_{\phi_F} [d\phi] e^{-S_E[\phi]} = \int_0 [d\phi] e^{-S_E[\phi+\overline{\phi}]} = \int_0 [d\phi] e^{-S_E[\phi+\overline{\phi}] - \Gamma'_E[T,\overline{\phi}]\phi} = e^{-\Gamma_E[T,\overline{\phi}]}$$

$\phi = \phi + \overline{\phi}$

$\overline{\phi} \to \phi_F, t \to \pm T/2$

$\Gamma'_E[T,\overline{\phi}] = 0$

Tunneling rate connected to effective action evaluated at extremum! (quantum bounce ◄—► Euclidean saddle point)

[Garbrecht & Millington]
[CT & A. Plascencia]

$$\gamma = \mathrm{Im}\left(\frac{-2}{VT_E}\Gamma_E[T,\bar{\phi}]\right), \quad \Gamma'_E[T,\bar{\phi}] = 0$$

Connection to the effective action

Analogous argument for Minkowski space-time

$$\gamma = \mathrm{Im}\left(\frac{2}{VT}\Gamma[T,\bar{\phi}]\right), \quad \Gamma'[T,\bar{\phi}] = 0$$

Can one evaluate Z[T] as a saddle point expansion around Minkowski bounce?

Picard-Lefschetz theory and complex saddle points

Real integrals as sum of complex steepest-descent paths

$$\int d^n x e^{\mathcal{I}(x)}$$

Cauchy theorem:

Contours along real x can be deformed to the complex plane
Interested in complex contours ending in convergence regions

$$h \equiv \mathrm{Re}\mathcal{I}(x) < 0$$

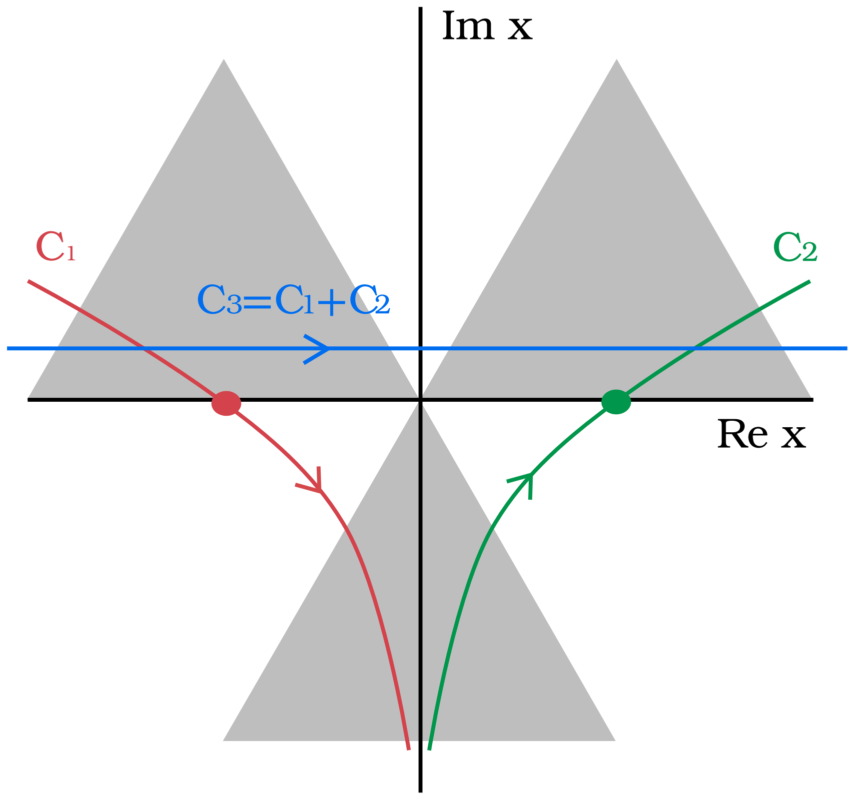
Picard-Lefschetz theory:

Can define basis of convergent contours
Elements of basis are steepest-descent paths ending in saddle points (which can be complex)

Integral = Sum over complex steepest-descent paths attached to saddle points

Real integrals as sum of complex steepest-descent paths

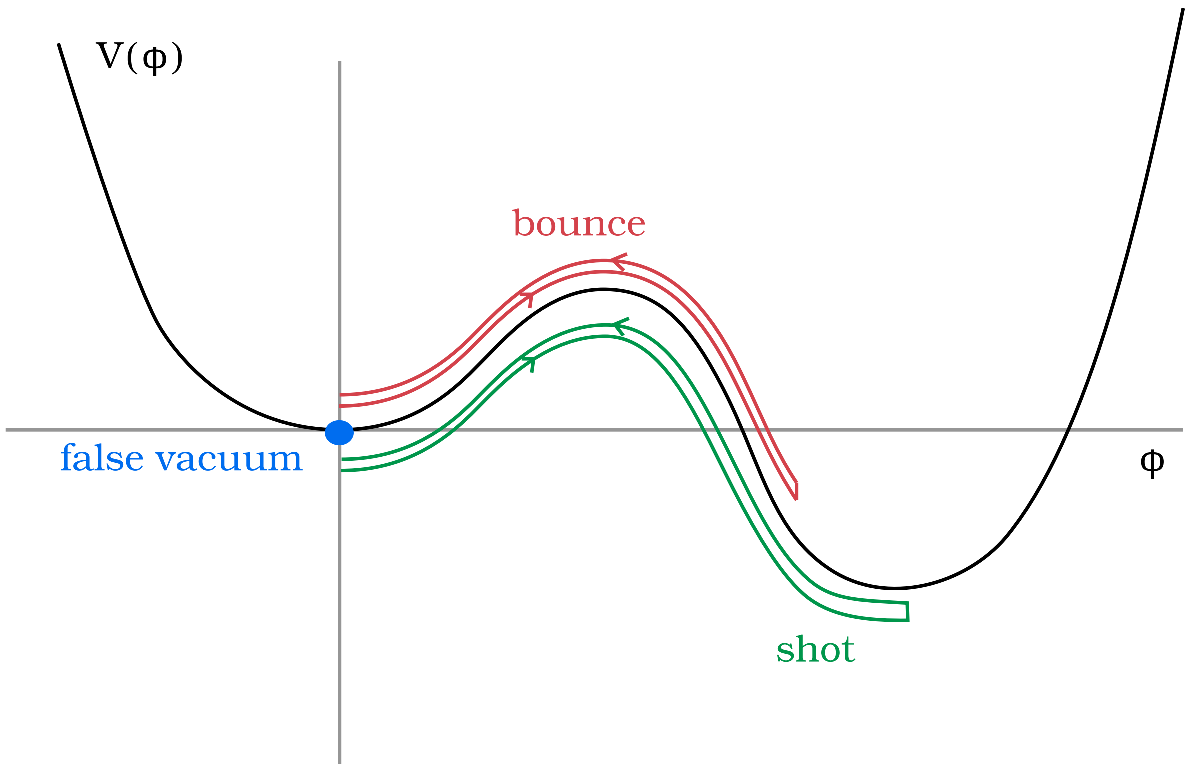
E.g. Airy function [Witten]: $Ai(\lambda) = \int dx e^{i\lambda\left(\frac{x^3}{3} - x\right)}, \quad \lambda > 0$ saddle points $x = \pm 1$



$$Ai(\lambda) = \int_{C_1 + C_2} dx e^{i\lambda\left(\frac{x^3}{3} - x\right)}$$

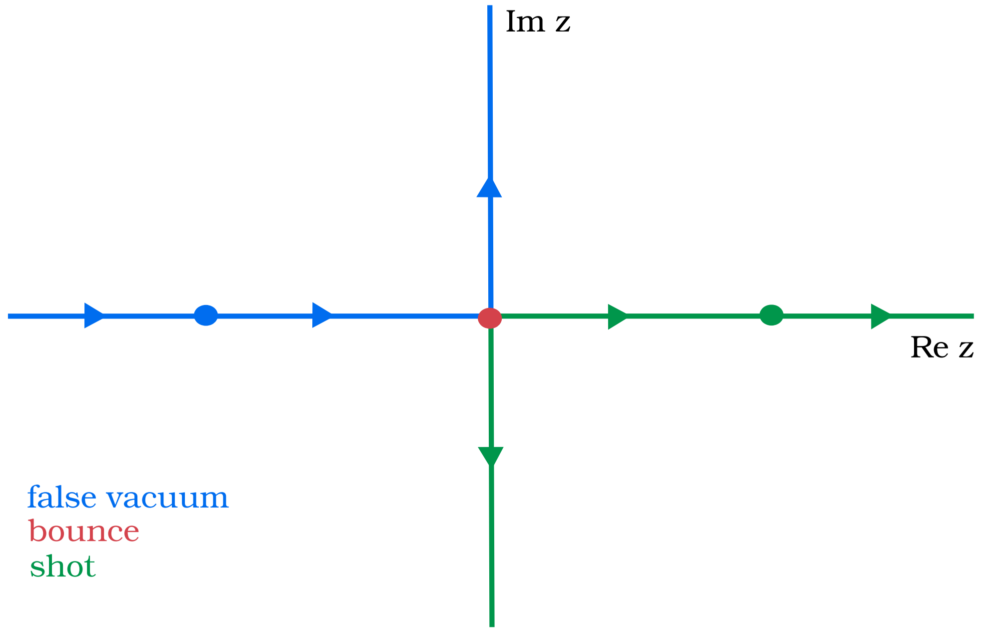
Steepest descent paths for the Euclidean integral

Euclidean action has 3 saddle points [Schwarz et al]



Steepest-descent paths for the Euclidean integral

Steepest-descent paths for a 1-parameter family of functions with parameter z



False-vacuum boundary conditions only from steepest-descent path connected to false vacuum! Z_E captures energy of true-vacuum state and is real. [Schwarz et al]

$$\langle q|e^{-HT_E}|q\rangle = \langle 0|e^{-HT_E}|0\rangle = Z_E[T_E] = Z_E^{\text{FV path}} + Z_E^{\text{shot path}} \in \mathbb{R}$$

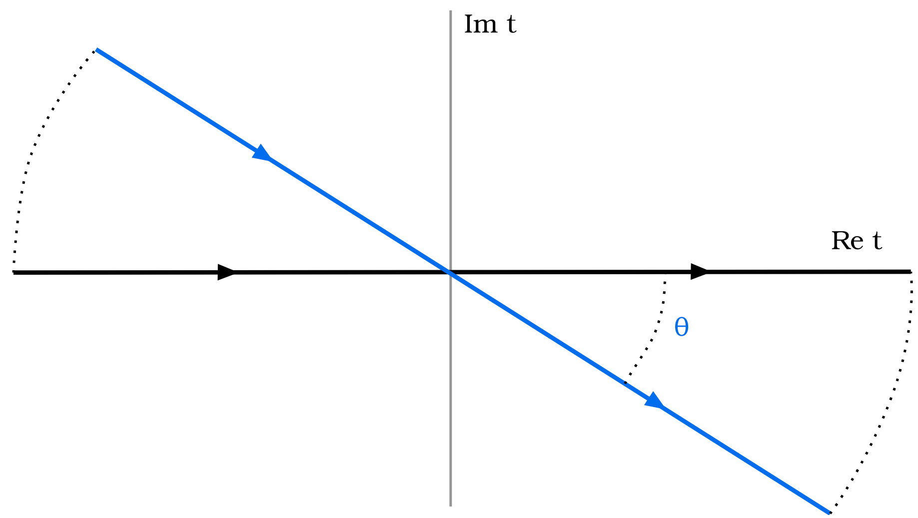
$$\langle F|e^{-HT_E}|F\rangle = Z_E^{FV}[T_E] = Z_E^{\text{FV path}} \sim Z_E^{\text{Gaussian, FV}} + \frac{1}{2} Z_E^{\text{Gaussian, bounce}}$$

One may relate decay rate to complex false-vacuum effective action [CT, A. Plascencia]

Minkowski bounce and fluctuation determinant

Choice of time-countours

We consider arbitrary rotations of the time contour away from Minkowski



$$S_\theta[\phi] = e^{-i\theta} \int d^4x \left[\frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 \cdot e^{+2i\theta} - \frac{1}{2} \left(\vec{\nabla}\phi \right)^2 - V(\phi) \right].$$

Path integral

$$\langle q | e^{-ie^{-i\theta} H T_\theta} | q \rangle = Z_\theta[T_\theta] = \int [d\phi] e^{iS_\theta[\phi]}$$

Complex saddle points

We complexify field-space and consider saddle-point equations

$$\left[e^{2i\theta}\frac{d^2}{dt^2} - \vec{\nabla}^2 + V'(\bar{\phi}(t,\vec{x}))\right]\bar{\phi} = 0.$$

Saddle points obtained by analytic continuation of Euclidean ones! (False vacuum, bounce, shot). Euclidean results recovered for $\theta = \frac{\pi}{2}$

$$\bar{\phi}(t,\vec{x}) = \bar{\phi}_E(\tau = ie^{-i\theta}t, \vec{x})$$

Bounce action at saddle related to that of the Euclidean case, as follows from Cauchy theorem

$$iS_\theta[\bar{\phi}] = ie^{-i\theta}\int d^4x \left[\frac{1}{2}\left(\frac{d\phi_E[ie^{-i\theta}t,\vec{x}]}{dt}\right)^2 \cdot e^{+2i\theta} - \frac{1}{2}\left(\vec{\nabla}\phi_E[ie^{-i\theta}t,\vec{x}]\right)^2 - V(\phi_E[ie^{-i\theta}t,\vec{x}])\right].$$

$$= -\int d\tau d^3x \left[\frac{1}{2}\left(\frac{d\phi_E[\tau,\vec{x}]}{d\tau}\right)^2 + \frac{1}{2}\left(\vec{\nabla}\phi_E[\tau,\vec{x}]\right)^2 + V(\phi_E[\tau,\vec{x}])\right] = -S_E[\bar{\phi}_E].$$

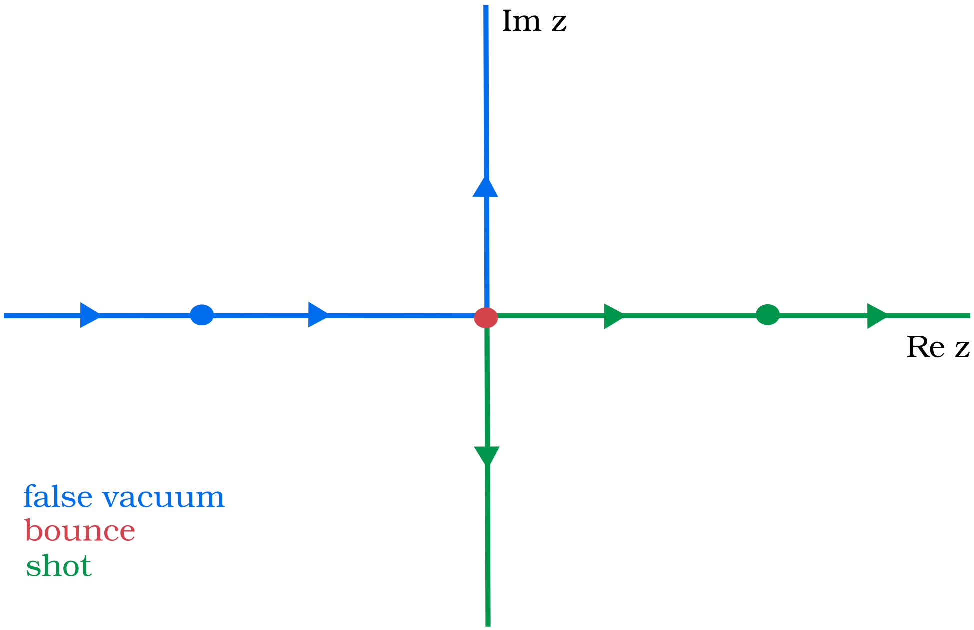
↑ $t \to -ie^{i\theta}\tau$

[Cherman & Unsal, Tanizaki & Koike]

Complex saddle points

False-vacuum path integral: As in Euclidean case, given by integration along steepest-descent path passing through false vacuum and bounce

$$\langle F|e^{-ie^{-i\theta}HT_\theta}|F\rangle = Z_\theta^{FV}[T_\theta] = Z_\theta^{\text{FV path}} \sim Z_\theta^{\text{Gaussian, FV}} + \frac{1}{2}Z_\theta^{\text{Gaussian, bounce}}$$



“Gaussian” contributions along steepest-descent paths

We expand to quadratic order around saddle points of the action and solve for the steepest-descent directions with parameter μ near false vacuum and bounce,

$$\phi = \overline{\phi} + \Delta\phi$$

$$Z_\theta[T_\theta] = \int [d\phi] e^{iS_\theta[\phi]} \sim e^{iS_\theta[\bar{\phi}]} \int [d\Delta\phi] e^{iS'_\theta[\bar{\phi},\Delta\phi]} = e^{-S_E[\bar{\phi}_E]} \int [d\Delta\phi] e^{iS'_\theta[\bar{\phi},\Delta\phi]}$$

Same exponential suppresion as in Euclidean space-time!
Need to match integration of fluctuations along steepest-descent path

$$\frac{d\Delta\phi(t,\vec{x},\mu)}{d\mu} = -\frac{\partial(iS'_\theta[\Delta\phi])^*}{\partial\Delta\phi^*(t,\vec{x},\mu)}$$

$$\frac{d(iS'_\theta)}{d\mu} = \frac{\partial(iS'_\theta)}{\partial\Delta\phi}\frac{d\Delta\phi}{d\mu} = -\left|\frac{\partial(iS'_\theta)}{\partial\Delta\phi}\right|^2 < 0$$

Ansatz [Yanizaki, Koike] $\Delta\phi_n = \sqrt{-i}e^{i\theta/2} g_n(\mu)\chi_n(t,\vec{x}), \quad g_n(\mu) \in \mathbb{R}$

$$\frac{dg_n(\mu)}{d\mu} = \kappa_n g_n(\mu), \quad \kappa_n > 0$$

$$\mathfrak{M}_\theta\, \chi_n(t,\vec{x}) \equiv \left(e^{2i\theta} \cdot \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + V''(\bar{\phi}(t))\right)\chi_n(t,\vec{x}) = \kappa_n \chi_n^*(t,\vec{x}).$$

“Gaussian” contributions along steepest-descent paths

Can be mapped to eigenvalue problem of hermitian operator

$$\begin{pmatrix} \mathbf{0} & \mathfrak{M}_\theta{}^* \\ \mathfrak{M}_\theta & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi_n \\ \chi_n^* \end{pmatrix} = \kappa_n \begin{pmatrix} \chi_n \\ \chi_n^* \end{pmatrix} .$$

Thus it is consistent to take κ_n real, and furthermore the χ_n are orthogonal.

$$Z_\theta^{\mathrm{Gaussian},\bar{\phi}} \sim e^{-S_E[\bar{\phi}_E]} \times \int [d\Delta\phi]\, e^{ie^{-i\theta}\int d^4x \left[-\frac{1}{2}\Delta\phi\, \mathfrak{M}_\theta \Delta\phi\right]} = e^{-S_E[\bar{\phi}_E]} \times \int [d\Delta\phi]\, e^{-\frac{1}{2}\sum_n \kappa_n g_n^2}$$

Integration measure along steepest-descent path

$$[d\Delta\phi] = \tilde{\mathcal{J}} \prod_n \frac{1}{\sqrt{2\pi}}\, dg_n .$$

Trade integration over zero modes with integration over collective coordinates

$$Z_\theta^{\mathrm{Gaussian},\bar{\phi}} \sim e^{-S_E[\bar{\phi}_E]} \times \sqrt{\frac{S_E[\bar{\phi}]}{2\pi}}(Vie^{-i\theta}T)\tilde{\mathcal{J}} \prod_n \frac{1}{\sqrt{\kappa_n}} .$$

"Gaussian" contributions along steepest-descent paths

$\tilde{\mathcal{J}} \prod_n \frac{1}{\sqrt{\kappa_n}}$ can be seen to be related to the usual determinant of $\mathfrak{M}_\theta$!

$$\begin{pmatrix} \mathbf{0} & {\mathfrak{M}_\theta}^* \\ \mathfrak{M}_\theta & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi_n \\ \chi_n^* \end{pmatrix} = \kappa_n \begin{pmatrix} \chi_n \\ \chi_n^* \end{pmatrix} \qquad \begin{pmatrix} \mathbf{0} & {\mathfrak{M}_\theta}^* \\ \mathfrak{M}_\theta & \mathbf{0} \end{pmatrix} \begin{pmatrix} i\chi_n \\ -i\chi_n^* \end{pmatrix} = -\kappa_n \begin{pmatrix} i\chi_n \\ -i\chi_n^* \end{pmatrix}$$

$$\prod_n (-{\kappa_n}^2) = \det \begin{pmatrix} \mathbf{0} & \mathfrak{M}_\theta^* \\ \mathfrak{M}_\theta & \mathbf{0} \end{pmatrix} = \det(-|\mathfrak{M}_\theta|^2) \quad \Longrightarrow \quad \prod_n \frac{1}{\sqrt{\kappa_n}} = |\det \mathfrak{M}_\theta|^{-1/2}$$

The integration measured is defined in the usual way in terms of real scalars.
$\tilde{\mathcal{J}}$ is the Jacobian relating the real scalars to the χ_n

$$\tilde{\mathcal{J}} = e^{i\mathrm{Arg}\ \det \mathfrak{M}_\theta/2}$$

$$Z_\theta^{\mathrm{Gaussian},\bar{\phi}}[T] = e^{-S_E[\bar{\phi}_E]} \sqrt{\frac{S_E[\bar{\phi}]}{2\pi}} (Vie^{-i\theta}T) e^{i\mathrm{Arg}\ \det \mathfrak{M}_\theta/2} |\det \mathfrak{M}_\theta|^{-1/2}$$

$$Z_E^{\mathrm{Gaussian},\bar{\phi}_E}[T_E] = e^{-S_E[\bar{\phi}_E]} \sqrt{\frac{S_E[\bar{\phi}]}{2\pi}} VT_E \left(-i|\det{}'\mathfrak{M}_E|^{-1/2}\right)$$

$$= -e^{-S_E[\bar{\phi}_E]} \sqrt{\frac{S_E[\bar{\phi}]}{2\pi}} VT_E \ e^{i\mathrm{Arg}\ \det \mathfrak{M}'_E/2} |\det{}'\mathfrak{M}_E|^{-1/2}$$

The final piece: the fluctuation determinant

If we manage to prove that the fluctuation determinants are related by straightforward analytic continuation

$$T_E \leftrightarrow ie^{-i\theta}T$$

Then we will have

$$Z_\theta^{\text{Gaussian},\bar{\phi}}[T] = Z_E^{\text{Gaussian},\bar{\phi}_{\text{E}}}[T_E]|_{T_E \to ie^{-i\theta}T}$$

exactly as follows from the formal definitions

$$Z_E[T_E] = \langle q|e^{-HT_E}|q\rangle \qquad Z_\theta[T] = \langle q|e^{-ie^{-i\theta}HT}|q\rangle$$

However, decay rates will only match if determinants become independent of T!

$$\gamma = \text{Im}\left(-\frac{1}{VT_E}Z_E^{\text{Gaussian, bounce}}[T_E]\right) = \text{Im}\, e^{-S_E[\bar{\phi}]}\sqrt{\frac{S_E[\bar{\phi}]}{2\pi}}\;\; e^{i\text{Arg det}\,\mathfrak{M}'_E/2}|\text{det}'\mathfrak{M}_E|^{-1/2}$$

$$\gamma = \text{Im}\left(-\frac{ie^{i\theta}}{VT}Z_\theta^{\text{Gaussian, bounce}}[T]\right) = \text{Im}\, e^{-S_E[\bar{\phi}]}\sqrt{\frac{S_E[\bar{\phi}]}{2\pi}}\;\; e^{i\text{Arg det}\,\mathfrak{M}'_\theta/2}|\text{det}'\mathfrak{M}_\theta|^{-1/2}$$

The strategy: determinant for rotated time

Determinant given in terms of eigenvalues of properly or improperly normalizable eigenfunctions

Solutions to the eigenvalue equations, regardless of normalizability, can be obtained by analytic continuation in time of Euclidean solutions (normalizable, or not)

One can characterize solutions by the parameters of their asymptotic form (large t)

Rotating parameters of Euclidean solutions and analytically continuing in time gives properly or improperly normalizable solutions for rotated time contours

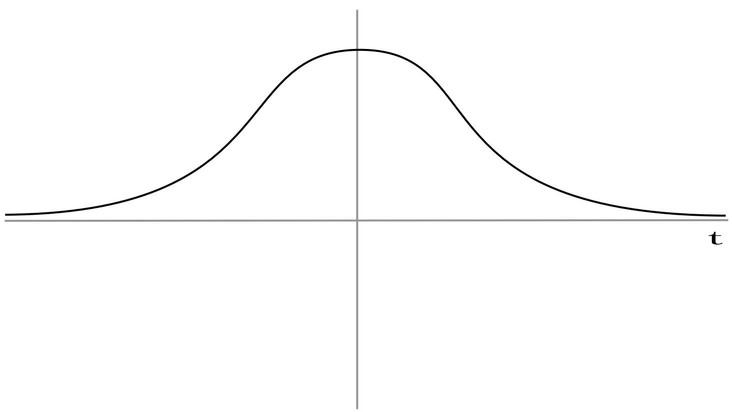
The determinant can be expressed in terms of the asymptotic parameters

Fluctuation equations and eigenfunctions

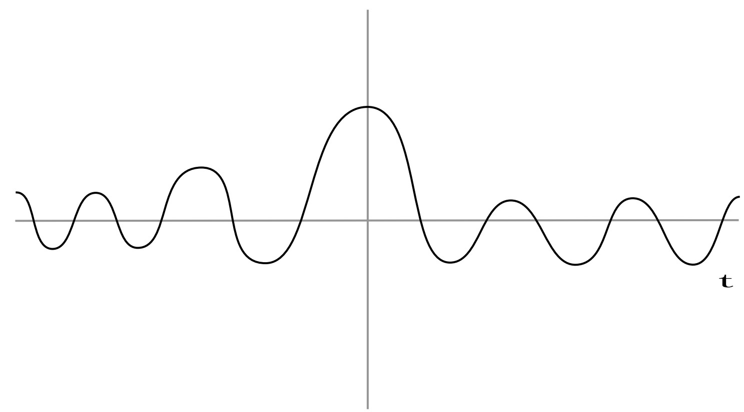
$$\mathfrak{M}_E\, \phi_E^\lambda(\tau,\vec{x}) \equiv \left[-\frac{d^2}{d\tau^2} - \vec{\nabla}^2 + V''(\bar{\phi}(\tau,\vec{x}))\right]\phi_E^\lambda(\tau,\vec{x}) = \lambda\phi_E^\lambda(\tau,\vec{x}),$$

$$\mathfrak{M}_\theta\, \phi^\lambda(t,\vec{x}) \equiv \left[e^{2i\theta}\frac{d^2}{dt^2} - \vec{\nabla}^2 + V''(\bar{\phi}(t,\vec{x}))\right]\phi^\lambda(t,\vec{x}) = \lambda\phi^\lambda(t,\vec{x}),$$

Euclidean operator real, self-adjoint: real, orthogonal eigenfunctions with real eigenvalues. Given Euclidean solution $\phi_E^\lambda(\tau,\vec{x})$ then $\phi^\lambda(t,\vec{x}) = \phi_E^\lambda(ie^{-i\theta}t,\vec{x})$ solves the rotated equation, but is it normalizable? Are rotated eigenfunctions orthogonal, are their eigenvalues real?



Temporally decaying
(discrete and continuum spectrum)



Temporally oscillating
(continuum spectrum)

Asymptotic solutions

At large time the equations simplify. Bounce tends to false vacuum and $V''(\bar{\phi}) \to m_F^2$

$$\phi^{\kappa,\vec{k}}(t,\vec{x}) \sim e^{-e^{-i\theta}\kappa\, t} e^{i\vec{k}\cdot\vec{x}}, \quad \lambda = \kappa^2 + \vec{k}^2 + m_F^2, \quad k^i, \kappa \in \mathbb{C}$$

Euclidean temporally decaying solutions generate temporally decaying solutions after analytic continuation $\tau \to i e^{-i\theta} t$ for $\theta \geq 0^+$

$$\phi_E^{i\beta,\vec{k}}(\tau,\vec{x}) \propto \exp\left(-\beta\tau\right), \ \tau \to \infty, \beta \in \mathbb{R}^+.$$

$$\phi_\theta^{i\beta,\vec{k}}(\tau,\vec{x}) \propto \exp\left(-ie^{-i\theta}\beta t\right) = \exp(-\sin\theta\beta t)\exp(-i\cos\theta\beta t), \ t \to \infty,$$

Same eigenvalue (same $\kappa, \vec{k}$)! Discrete spectrum preserved under time rotations

Euclidean temporally oscillating solutions generate oscillating solutions only after appropriate complex rotations of $\kappa_E \to -ie^{i\theta}\kappa_\theta$

$$\phi_E^{\kappa,\vec{k}}(\tau,\vec{x}) \propto \exp\left(i\kappa\tau\right), \ \tau \to \infty, \qquad \lambda_E = \kappa^2 + \vec{k}^2 + m_F^2$$

$$\phi_\theta^{-ie^{i\theta}\kappa_\theta,\vec{k}}(t,\vec{x}) \propto \exp\left(i\kappa_\theta\tau\right), \ \tau \to \infty, \qquad \lambda_\theta = -e^{2i\theta}\kappa_\theta^2 + \vec{k}^2 + m_F^2$$

Continuous spectrum changes under time rotations!

The determinant

Despite non-hermiticity of $\mathfrak{M}_\theta$, eigenfunctions remain orthogonal (follows from orthogonality of Euclidean solutions and Cauchy theorem)

$$\int \phi^{\kappa_n,\vec{k}_n}\phi^{\kappa_m,\vec{k}_m} = \delta_{mn}$$

Discrete spectrum (No conjugation!)

$$\int \phi^{-ie^{i\theta}\kappa,\vec{k}}\phi^{-ie^{i\theta}\kappa',\vec{k}'} = \delta(\kappa-\kappa')\delta^{(3)}(\vec{k}-\vec{k}')$$

Continuum spectrum

$\mathfrak{M}_\theta$ can then be expanded in terms of projection operators

$$\mathfrak{M}_\theta(x,x') = \sum_{disc}\lambda(\kappa,\vec{k})\,\phi^{\kappa,\vec{k}}(x)\phi^{\kappa,\vec{k}}(x') + \int_{cont} d\kappa d^3k_\theta\,\lambda(\theta,\kappa_\theta,\vec{k})\,\phi^{(\kappa_\theta,\vec{k})}(x)\phi^{(\kappa_\theta,\vec{k})}(x')$$

From which it follows

$$\log\det\mathfrak{M}_\theta = \mathrm{tr}\,\log\mathfrak{M}_\theta = \sum_{disc}\log\lambda + \hat{V}_\theta^{(4)}\int_{cont}\frac{d\kappa_\theta d^3k}{(2\pi)^4}\log\lambda(\theta,\kappa_\theta,\vec{k})$$

$$= \sum_{disc}\log\lambda + \hat{V}_\theta^{(4)}\int_{cont}\frac{d\kappa_\theta d^3k}{(2\pi)^4}\log(e^{2i\theta}\kappa_\theta^2+\vec{k}^2+m_F^2)$$

$$\hat{V}_\theta^{(4)} = TV\,(+\text{finite})$$

The determinant

$$\log\det\mathfrak{M}_\theta = \sum_{dec,disc}\log\lambda + \hat{V}_\theta^{(4)}\int_{osc}\frac{d\kappa_\theta d^3k}{(2\pi)^4}\log(-e^{2i\theta}\kappa_\theta^2 + \vec{k}^2 + m_F^2)$$

$$\log\det\mathfrak{M}_E = \sum_{dec,disc}\log\lambda + \hat{V}_E^{(4)}\int_{osc}\frac{d\kappa_E d^3k}{(2\pi)^4}\log(\kappa_E^2 + \vec{k}^2 + m_F^2)$$

Rotating the κ_θ contour to $\kappa_\theta \to ie^{-i\theta}\kappa_E$ and ignoring the "finite" in $\hat{V}_\theta^{(4)} = TV + \text{finite}$, then using the Cauchy theorem we would get the desired result for equivalence of partition functions:

$$\det\mathfrak{M}_\theta = \det\mathfrak{M}_\theta|_{T_E=ie^{i\theta}T} \Rightarrow Z_\theta^{\text{Gaussian}}[T] = Z_E^{\text{Gaussian}}[T_E]|_{T_E\to ie^{-i\theta}T}$$

However, note the determinants are dependent on T, which as seen before precludes invariance of decay rate.

But there is a way out: contribution of continuum spectrum is proportional to Coleman-Weinberg potential evaluated at false vacuum (assumed zero vac. at tree level)!

$$\int_{osc}\frac{d\kappa_E d^3k}{(2\pi)^4}\log(\kappa_E^2 + \vec{k}^2 + m_F^2) = -2V_{CW}(\phi_F)$$

Ensuring that the effective potential at the false vacuum is zero gives consistent partition functions and decay rates

The need for such requirement had already been argued before, on the basis of the relation of tunneling rates to a false-vacuum effective action evaluated at bounce solution [CT, A. Plascencia]

An optical theorem for the vacuum

$$\langle F|e^{-iHT}|F\rangle = Z_0^{\text{Gaussian, FV}} + \frac{1}{2}Z_0^{\text{Gaussian, bounce}}$$

For the false vacuum saddle: the fluctuation operator has no zero modes, the eigenfunctions are plane waves with a continuum spectrum

$$Z_0^{\text{Gaussian, FV}} = e^{-iVTV_{CW}(\phi_F)} = 1$$

We may call $\frac{1}{2}Z_0^{\text{Gaussian, bounce}} \equiv i\mathcal{M}$ ⟹ $\langle F|e^{-iHT}|F\rangle = 1 + i\mathcal{M}$

Total decay probability

$$VT\gamma = -\text{Im}\left(iZ_0^{\text{Gaussian, bounce}}[T]\right) = 2\text{Im}\mathcal{M}$$

Coincides with unitarity requirement for scattering operator

$$\langle F|e^{-iHT}|F\rangle = 1 + i\mathcal{M} = \langle F|\hat{\mathcal{S}}|F\rangle = \langle F|\mathbb{I} + i\hat{\mathcal{M}}|F\rangle$$

$$\hat{\mathcal{S}}\hat{\mathcal{S}}^\dagger = \mathbb{I} \Rightarrow 2i(\hat{\mathcal{M}} - \hat{\mathcal{M}}^\dagger) = \hat{\mathcal{M}}\hat{\mathcal{M}}^\dagger$$

An optical theorem for the vacuum

$$\langle F|2i(\hat{\mathcal{M}}-\hat{\mathcal{M}}^\dagger)\rangle = \langle F|\hat{\mathcal{M}}\hat{\mathcal{M}}^\dagger|F\rangle$$

$$2\mathrm{Im}\mathcal{M} = \sum_Q \langle F|\hat{\mathcal{M}}|Q\rangle\langle Q|\hat{\mathcal{M}}^\dagger|F\rangle = \text{Total decay probability}$$

The requirement of zero effective potential at the minimum is equivalent to a proper normalization of the vacuum

$$\langle F|e^{-iHT}|F\rangle = \langle F|\mathbb{I}+i\hat{\mathcal{M}}|F\rangle = 1+i\mathcal{M}$$ is assuming $$\langle F|F\rangle = 1$$

and as seen before it implies $V_{CW}(\phi_F) = 0$

Conclusions

Tunneling rates can be consistently calculated in Minkowski space-time, or for any arbitrarily rotated time contour

Consistency of results requires the full effective potential to be zero at the false vacuum

This allows to recover the optical theorem

Vacuum energy in flat space can be thought as being fixed to zero by demanding a proper normalization of the vacuum state